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An Interesting Application of the Difference of Two Squares

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A standard topic in a college algebra course is solving equations containing two radicals. In this paper, we present an interesting application of the difference of two squares to solve such an equation.

Introduction

We have been teaching a college algebra course for a number of years. After discussing how to solve equations containing one radical, a standard topic is solving an equation that contains two radicals, usually two square roots. The traditional method to solve such an equation is to isolate one of the square roots and then square both sides. We simplify both sides of the equation by combining like terms and obtaining an equation that contains only one square root. We then isolate that square root and square the resulting equation again to obtain the final solution by isolating the variable on one side. Thus this technique requires squaring a given equation twice to reach the final solution. In general, our students find it difficult to properly and accurately carry out the computations involved in this process.

In this paper we show an interesting way of using the difference of two squares to solve equations containing two radicals. We have not found this method presented in the literature elsewhere. It was surprising for us considering the simplicity of the approach. By squaring the equation only once, this method simplifies the computations and the calculation of the final solution. In our presentation, we have judiciously chosen examples to avoid fractions. Our emphasis is on explaining the technique rather than getting bogged down in mundane calculations.

The main advantage of this technique is that it can be used to solve more difficult problems without any extra effort.

Example 1

Solve: $\sqrt{x-2}+\sqrt{x-5}=6$ (1)

Create: $(x-2)-(x-5)=3$ (2)

We obtain the left side of equation (2) by simply keeping the same expressions that are under the radicals on the left side of equation (1) and placing a minus sign between them. Simplifying the expression on the left side of equation (2) results in the right-hand side of equation (2), making equation (2) an identity. Equation (2) always contains a minus sign between the parentheses so that it can be factored out using the difference of two squares. Using the difference of two squares, we first factor the left hand side of equation (2).

$$(x-2)-(x-5) = \left(\sqrt{x-2}-\sqrt{x-5}\right)\left(\sqrt{x-2}+\sqrt{x-5}\right)$$

Both factors contain radicals, one of them the same as the left hand side of equation (1). Dividing the factored form of left side of equation (2) by the left side of equation (1) and cancelling out the common factor and, moreover, dividing the right side of equation (2) by the right side of equation (1), we obtain the following equation

$\sqrt{x-2}-\sqrt{x-5}=0.5$ (3)

We have not found this method presented in the literature elsewhere. It was surprising for us considering the simplicity of the approach.

Adding equations (1) and (3), we obtain $2\sqrt{x-2} = 6.5$. Squaring both sides, we get $4(x-2) = 42.25$ or $(x-2) = 10.5625$. Thus $x = 12.5625$. Substituting 12.5625 back into the original equation for x, we find that 12.5625 is a correct solution.

One can easily use letters instead of numbers to see the solution in general terms. We show this in our next example.

Note: In the following examples, we continue to use the same method as in Example 1 above i.e., factoring the left side of equation (2) using the difference of two squares, dividing the factored left side of equation (2) by the left side of equation (1) and the right side of equation (2) with the right side of equation (1) to obtain equation (3). However, we will refer to it as dividing equation (2) by equation (1) in the rest of our examples.

We first transfer one of the radicals to the right/left hand side, and after squaring the first time, we are only left with one of the two original radical terms.

Example 2

Solve: $\sqrt{x+a} \pm \sqrt{x+b} = c \qquad (1)$

Create: $(x+a) - (x+b) = a - b \qquad (2)$

Dividing equation (2) by equation (1) we get

$$\sqrt{x+a} \pm \sqrt{x+b} = \tfrac{(a-b)}{c} (c \neq 0). \qquad (3)$$

Adding equations (1) and (3) we obtain

$$2\sqrt{x+a} = \tfrac{(a-b)}{c} + c$$

or $2c\sqrt{x+a} = (a-b) + c^2$.

Squaring we get

$$4c^2(x+a) = (a-b+c^2)^2$$

or $(x+a) = \frac{(a-b+c^2)^2}{4c^2}$.

Thus $x = \frac{(a-b+c^2)^2}{4c^2} - a$.

Example 2 provides us with a formula to solve a radical equation of the form $\sqrt{x+a} \pm \sqrt{x+b} = c$. More importantly, it shows that the set of all possible solutions to both types of equations

$$\sqrt{x+a} \pm \sqrt{x+b} = c$$

is the same; however, they may still have different solutions. This is shown in Examples 3 and 4. Also note that the solution 12.5625 from Example 1 does not solve $\sqrt{x-2} - \sqrt{x-5} = 6$.

From Examples 1 and 2, it appears that we always retain the first square root term and the second square root term vanishes. This, however, is not the case because if we subtract (instead of adding) equation (3) from equation (1) in the above examples, we will retain the second square root term and the rest of the solution will be similar. It is, in fact, in line with the traditional method in which we first transfer one of the radicals to the right/left hand side, and after squaring the first time, we are only left with one of the two original radical terms.

We now present an example in which our second equation results in an algebraic expression instead of a constant term. We picked problem #1 from page 131 of Blitzer's (2001) *College Algebra* text.

Example 3

Solve: $\sqrt{3x+1} + \sqrt{x+4} = 1 \qquad (1)$

Create: $(3x+1) - (x+4) = 2x - 3 \quad (2)$

Dividing equation (2) by equation (1), we get

$$\sqrt{3x+1} + \sqrt{x+4} = 2x - 3. \qquad (3)$$

Adding equations (1) and (3), we obtain $2\sqrt{3x+1} = 2x - 2$. Thus, $\sqrt{3x+1} = x - 1$. Squaring both sides, we get $3x + 1 = x^2 - 2x + 1$ or $x^2 - 5x = 0$ (i.e., $x(x-5) = 0$). Thus, $x = 5$ or $x = 0$.

Neither of the two answers checks. In other words, neither solution when plugged back into our original equation gives us a true statement. Thus this equation has no solution.

Our next example is almost the same as Example 3, except that there is a change in sign between the two radicals. The consequences of changing the sign are discussed after presenting the solution to the example.

Example 4

Solve: $\sqrt{3x+1} - \sqrt{x+4} = 1$ (1)
Create: $(3x+1) - (x+4) = 2x - 3.$ (2)

Dividing equation (2) by equation (1), we get

$$\sqrt{3x+1} - \sqrt{x+4} = (2x-3). \quad (3)$$

Adding equations (1) and (3), we obtain
$2\sqrt{3x+1} = 2x - 2.$

Thus, $\sqrt{3x+1} = x - 1$. Squaring both sides, we get $3x+1 = x^2 - 2x + 1$ or $x^2 - 5x = 0$ (i.e., $x(x-5) = 0$). Thus, $x = 5$ or $x = 0$. The solution $x = 5$ checks in the equations but $x = 0$ does not. Thus x = 5 is the only solution.

Equations in Examples 3 and 4 differ only in a sign between the two radical expressions; however, both equations have the same set of possible solutions. While Example 3 has no solution, Example 4 does have a solution. These examples emphasize for our students that the change of a sign in mathematics can have significant consequences.

The general form of the equations in Examples 3 and 4 above is $\sqrt{ax+b} \pm \sqrt{cx+d} = e$. We look at one of these in our next example.

These examples emphasize for our students that the change of a sign in mathematics can have significant consequences.

Example 5

Solve: $\sqrt{ax+b} + \sqrt{cx+d} = e$ (1)
Create:
$(ax+b) - (cx+d) = (a-c)x + (b-d)$ (2)

Dividing equation (2) by equation (1), we get

$$\sqrt{ax+b} - \sqrt{cx+d} = \frac{(a-c)x+(b-d)}{e}, (e \neq 0). \quad (3)$$

Adding equations (1) and (3), we obtain
$2\sqrt{ax+b} = \frac{(a-c)x+(b-d)}{e} + e$

Thus, $2e\sqrt{ax+b} = (a-c)x + (b-d) + e^2$. For notational convenience, let $P = a - c$ and $Q = (b-d) + e^2$, then we can write the last equation as $2e\sqrt{ax+b} = Px + Q$. Squaring both sides, we get $4e^2(ax+b) = (Px+Q)^2$. The resulting quadratic equation can be solved in terms of the constants *a*, *b*, *c*, *d*, and *e*. As our final example, we have picked an example that most of our college algebra students may not even dare to try.

Example 6

Solve: $\sqrt{2x^2-5x+3}+\sqrt{2x^2-x+1}=2$ (1)
Create: $(2x^2-5x+3)-(2x^2-x+1)=2-4x$ (2)

Dividing equation (2) by equation (1), we get
$\sqrt{2x^2-5x+3}-\sqrt{2x^2-x+1}=1-2x$. (3)

Adding equations (1) and (3), we obtain
$2\sqrt{2x^2-5x+3}=3-2x$.

Squaring both sides, we get
$4(2x^2-5x+3)=9-12x+4x^2$ or $4x^2-8x+3=0$ (i.e., $(2x-3)(2x-1)=0$).

Thus, $x=\frac{3}{2}$ or $x=\frac{1}{2}$. However, for the equation $\sqrt{2x^2-5x+3}+\sqrt{2x^2-x+1}=2$, we get the same set of solutions; but only $x=\frac{1}{2}$ solves the equation. Thus this equation has only one solution.

In our final example, Example 6, we have deliberately kept the coefficient of x^2 to be the same in both the radicals. Otherwise, when we square both sides, we will obtain a fourth degree equation which might be difficult to solve, especially at the college algebra level. The equation may not even be solvable using simple algebraic techniques.

The final answer(s) for the two equations may be quite different. This could be easily overlooked using the traditional method.

Conclusions

The application of the difference of two squares to solve an equation containing two radicals, as presented in this paper, may put our students at ease when solving such equations as they will not have to square the equation twice in order to obtain the final solution. It also gives them an alternative to the traditional method of solving equations with radicals. Another interesting observation is that while the two types of radical equations (one with a plus sign and another with a minus sign)

$$\sqrt{x+a}\pm\sqrt{x+b}=c$$

both have the same set of possible solutions (Examples 3 and 4 above), the final answer(s) for the two equations may be quite different. This could be easily overlooked using the traditional method. This method also helps solve radical equations that may otherwise seem too challenging for our students to attempt (e.g., the equation in Example 6 above). One of the stated principles of NCTM (2000) is “Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well” (p. 28). This technique allows helps students explore two concepts further: radicals and the difference of two squares, thus allowing them to gain a better understanding of each. Finally, this article may provide motivation and some fuel for our students to do undergraduate research in mathematics. For example, students can be challenged to mathematically support the rationale for manipulations of each side of the equations.

References

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